EL3370 Mathematical Methods in Signals, Systems and Control

Topic 10: Application to H_∞ Control Theory

Cristian R. Rojas

Division of Decision and Control Systems
KTH Royal Institute of Technology

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

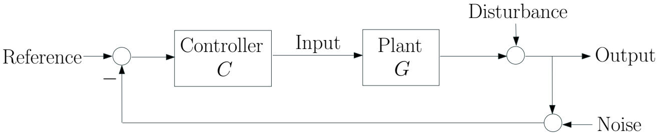
Nehari's Theorem

H_∞ Control Example

Bonus Slides

Feedback Control and Youla Parameterization

Goal: Design a controller that drives the output as close to the reference as possible.



Concerns:

1. Reference: Output should be equal to reference.
2. Disturbance: Disturbance should not affect output.
4. Noise: Noise should not perturb output.
5. Input: Input should lie within prescribed limits.
6. Stability: Closed loop should be stable.
7. Robustness: Model errors should not affect performance nor stability.

Feedback Control and Youla Parameterization (cont.)

Reminder: If $x = (x[k])_{k\in\mathbb{N}_0}$ is a real sequence, its *Z-transform* is

$$X(z) := \mathcal{Z}\{x\}(z) := \sum_{k=0}^{\infty} x[k] z^{-k},$$

where z is restricted to the subset of $\mathbb{C}$ where the sum is convergent.

If $\mathcal{Z}\{\text{ref.}\} =: R(z)$, $\mathcal{Z}\{\text{noise}\} =: N(z)$, $\mathcal{Z}\{\text{disturb.}\} =: D(z)$, $\mathcal{Z}\{\text{in.}\} =: U(z)$ and $\mathcal{Z}\{\text{out.}\} =: Y(z)$:

$$\left.\frac{Y(z)}{R(z)}\right|_{D,N=0} = \frac{G(z)C(z)}{1+G(z)C(z)} =: T(z) \qquad (\textit{complementary sensitivity})$$

$$\left.\frac{Y(z)}{D(z)}\right|_{R,N=0} = \frac{1}{1+G(z)C(z)} = 1 - T(z) =: S(z) \qquad (\textit{sensitivity})$$

$$\left.\frac{Y(z)}{U(z)}\right|_{D,N=0} = \frac{G(z)}{1+G(z)C(z)} =: S_i(z) \qquad (\textit{input sensitivity})$$

$$\left.\frac{U(z)}{R(z)}\right|_{D,N=0} = \frac{C(z)}{1+G(z)C(z)} =: S_u(z) \qquad (\textit{control sensitivity})$$

A control loop is *internally stable* if all these sensitivities are stable.

Feedback Control and Youla Parameterization (cont.)

Many of the concerns can be traded-off by imposing, *e.g.*, that

- $T(e^{i\omega}) \approx 1$ for small ω,
- $T(e^{i\omega}) \approx 0$ for large ω,
- the closed loop is internally stable.

This can be achieved by requiring that C yields a stable closed loop and minimizes

$$\|W_1(1-T)\|_\infty + \|W_2 T\|_\infty = \sup_{|z|=1} |W_1(z)[1-T(z)]| + \sup_{|z|=1} |W_2(z)T(z)|. \qquad (W_1, W_2: \text{weights})$$

To parameterize all stabilizing controllers C, the following result is useful:

Theorem (Youla/affine parameterization) (*see bonus slides for proof*)
Assume that G is stable. Then C yields an internally stable loop iff the *Youla parameter* $Q := C/(1+GC)$ is stable. Furthermore, all sensitivity functions are affine functions of Q.

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

Approaches to H_∞ Control

(a) **Nehari problem (H_∞ approximation)** $\Leftarrow$ we will follow this approach!

(b) Nevanlinna-Pick problem (H_∞ interpolation)

(c) Polynomial methods (H. Kwakernaak)

(d) Chain scattering (H. Kimura)

(e) Riccati equations (“DGKF” paper)

(f) Linear matrix inequalities (P. Gahinet & P. Apkarian, C. Scherer)

(g) Differential games (T. Başar and P. Bernhard)

(h) Krein space techniques

$\vdots$

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

The Big Picture

Our goal is to obtain the minimizer, over all $Q \in H_\infty$, of $\|T - GQ\|_\infty$, where $T \in L_\infty(\mathbb{T})$ (recall from Topic 2 that $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$) and $G \in H_\infty$. Now,

$$
\begin{aligned}
\min_{Q \in H_\infty} \|T - GQ\|_\infty &= \min_{\tilde{Q} = G_O Q \in H_\infty} \alpha \|G_I^{-1} T - \tilde{Q}\|_\infty \quad (G = G_I G_O, \text{ where } G_O, G_O^{-1} \in H_\infty, |G_I(e^{i\omega})|^2 = \alpha^2 = \text{constant}) \\
&= \min_{\tilde{Q} = G_O Q \in H_\infty} \alpha \left\| [G_I^{-1} T]_{\text{stable}} + [G_I^{-1} T]_{\text{unstable}} - \tilde{Q} \right\|_\infty \\
&= \min_{Q' = \tilde{Q} - [G_I^{-1} T]_{\text{stable}}} \alpha \left\| [G_I^{-1} T]_{\text{unstable}} - Q' \right\|_\infty, \text{ where } Q' \in H_\infty, [G_I^{-1} T]_{\text{unstable}} \in H_\infty^\perp \\
&= \alpha \left\| \Gamma_{[G_I^{-1} T]_{\text{unstable}}} \right\|, \text{ where } \Gamma_{[G_I^{-1} T]_{\text{unstable}}} \text{ is a } \textit{Hankel} \text{ operator.} \quad \text{(Nehari's theorem)}
\end{aligned}
$$

In this topic, we will define the appropriate H_p spaces, the *inner-outer factorization* ($G = G_I G_O$), Hankel operators, Nehari's theorem, and how to compute the minimizer!

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

Hardy Spaces

Definition

For $1 \leqslant p < \infty$, the *Hardy space* H_p is the normed space of analytic functions f on the exterior of the unit disc, $\mathbb{E} := \{z \in \mathbb{C} \colon |z| > 1\}$, for which the norm

$$\|f\|_p := \sup_{1<r\leqslant\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\omega})|^p \, d\omega \right)^{1/p}$$

is finite. H_∞ is the space of bounded analytic functions f on $\mathbb{E}$, with norm

$$\|f\|_\infty := \sup_{z\in\mathbb{E}} |f(z)| = \sup_{\substack{-\pi\leqslant\omega<\pi \\ 1<r\leqslant\infty}} |f(re^{i\omega})|.$$

Remark. For $1 \leqslant p < q \leqslant \infty$, $H_p \supseteq H_q$: indeed, for fixed $r \in (1,\infty]$, with $f_r(\omega) := f(re^{i\omega})$, so $f_r \in L_q[-\pi,\pi]$; Hölder's inequality yields $\int_{-\pi}^{\pi} |f(re^{i\omega})|^p \, d\omega = \|f_r\|_p^p = \|1 \cdot f_r^p\|_1 \leqslant \|1\|_{q/(q-p)} \|f_r^p\|_{q/p} = (2\pi)^{1-p/q} \|f_r\|_q^p$, *i.e.*, $\|f_r\|_p \leqslant (2\pi)^{1/p-1/q} \|f_r\|_q$. In particular, $H_\infty \subseteq H_2 \subseteq H_1$.

We can identify elements of H_p with functions in $L_p(\mathbb{T})$! (recall that $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$).

Theorem. For every $f \in H_p$ ($1 \leqslant p \leqslant \infty$) the *radial limit* $\tilde{f}(e^{i\omega}) = \lim_{r \to 1_+} f(re^{i\omega})$ exists for almost every $\omega \in [-\pi, \pi]$, and indeed $\tilde{f} \in L_p(\mathbb{T})$, with $\|\tilde{f}\|_{L_p} = \|f\|_{H_p}$.

(*See bonus slides for proof in the case* $1 < p \leqslant \infty$)

Remark

H_p can be identified with a closed subspace of $L_p(\mathbb{T})$, and hence it is a Banach space. Indeed, H_p can be defined as the subspace of those $f \in L_p(\mathbb{T})$ whose negative Fourier coefficients vanish, *i.e.*, $f(e^{i\omega}) = \sum_{n=-\infty}^{\infty} a_n e^{-in\omega}$ with

$$a_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\omega})\, e^{in\omega} d\omega = 0 \quad \text{for } n < 0.$$

Those f's can be extended to $\mathbb{E}$ as $f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ for $z \in \mathbb{E}$.

Hardy Spaces (cont.)

In particular, H_2 is a Hilbert space, since it is a closed subspace of $L_2(\mathbb{T})$, and we can define the *projection* operator from $L_2(\mathbb{T})$ onto H_2 as

$$P_{H_2}: \sum_{n=-\infty}^{\infty} a_n e^{-in\omega} \mapsto \sum_{n=0}^{\infty} a_n e^{-in\omega}.$$

H_2 can also be identified with ℓ_2, by: $\quad \omega \mapsto \sum_{n=0}^{\infty} a_n e^{-in\omega} \in H_2 \quad \Leftrightarrow \quad (a_0, a_1, \ldots) \in \ell_2.$

Note. H_p with $p \neq 2$ *cannot* be identified with ℓ_p.

$H_2^{\perp}$ is the orthogonal complement of H_2 in $L_2(\mathbb{T})$, *i.e.*, $f \in H_2^{\perp}$ iff it has the form $f(e^{i\omega}) = \sum_{n=-\infty}^{-1} a_n e^{-in\omega}$.

RH_p and RL_p are those subspaces of H_p and $L_p(\mathbb{T})$ consisting of those functions which are *real-rational* (*i.e.*, quotients of polynomials with real coefficients).

Hardy Spaces (cont.)

For some derivations, we will need the following technical lemma:

Lemma. If $f \in H_2 \setminus \{0\}$, then $f(e^{i\omega}) \neq 0$ almost everywhere, and $\int_{-\pi}^{\pi} \log|f(e^{i\omega})| d\omega > -\infty$.

Proof (Helson and Lowdenslager, 1958)
If $f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ is non-zero, by multiplying it by some z^m ($m \in \mathbb{N}$) we assume w.l.o.g. that $a_0 \neq 0$.

Consider the affine subspace $C = \{z \mapsto f(z)[1 + b_1 z^{-1} + \cdots + b_m z^{-m}]\colon m \in \mathbb{N}; b_1, \ldots, b_m \in \mathbb{C}\} \subseteq H_2$; note that $0 \notin \bar{C}$, since if $h \in C$, $h(\infty) = a_0 \neq 0$. By the closest point property, there is a $g \in \bar{C}$ of smallest norm.

Given $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, $\|g + \lambda z^{-m} g\|^2 = (1 + |\lambda|^2)\|g\|^2 + 2\mathrm{Re}\left[(\lambda/2\pi i)\int_{-\pi}^{\pi} |g(e^{i\omega})|^2 e^{-im\omega} d\omega\right]$, but since $g + \lambda z^{-m} g \in \bar{C}$ and g has minimum norm in $\bar{C}$, $\int_{-\pi}^{\pi} |g(e^{i\omega})|^2 e^{-im\omega} d\omega = 0$ for all $m \in \mathbb{N}$, and taking the conjugate the same holds for all $-m \in \mathbb{N}$; thus, $|g(e^{i\omega})|^2 \equiv g_0 > 0$, since $g \neq 0$.

Assume $f(z) = 0$ on a set $E \subseteq \mathbb{T}$. Define $h\colon \mathbb{T} \to \mathbb{C}$ as $h(z) = 0$ on $\mathbb{T} \setminus E$, and $h(z) = |g(z)|/\overline{g(z)}$ on E. Then, $h \in L_2(\mathbb{T})$ and $(F, h) = 0$ for all $F \in C$ (since F also vanishes on E), and by continuity, $(F, h) = 0$ for all $F \in \bar{C}$, so $0 = (g, h) = (2\pi)^{-1} \int_E |g(e^{i\omega})| d\omega = (2\pi)^{-1}\sqrt{g_0}\, m(E)$ (where m is the Lebesgue measure), hence E has measure zero.

Now, for $\varepsilon > 0$, let $\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[|f(e^{i\omega})|^2 + \varepsilon] d\omega$ and $\psi = \lambda - \log[|f|^2 + \varepsilon]$. Then, since $\int_{-\pi}^{\pi} \psi(e^{i\omega}) d\omega = 0$, e^{ψ} can be approximated arbitrarily well in $\mathbb{T}$ by polynomials of the form $|1 + b_1 z^{-1} + \cdots + b_m z^{-m}|^2$ (recall Topic 5), so

$$\exp\left\{\frac{1}{2\pi}\int \log[|f|^2 + \varepsilon]\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\lambda) d\omega = \frac{1}{2\pi}\int e^{\psi}(|f|^2 + \varepsilon) \geqslant \frac{1}{2\pi}\int e^{\psi}|f|^2 \geqslant \inf_{F \in \bar{C}} \|F\|^2 = g_0 > 0.$$

The monotone convergence theorem, for $\varepsilon \to 0$, yields $\int_{-\pi}^{\pi} \log|f(e^{i\omega})|^2 d\omega > -\infty$. $\square$

Inner-Outer Factorization

Example:

$$4\frac{(z-2)(z-3)}{(z-0.5)(z-0.6)} = \underbrace{\frac{(z-2)(z-3)}{(1-2z)(1-3z)}}_{\substack{\text{“inner function”} \\ \text{(constant modulus = 1 in } \mathbb{T})}} \cdot \underbrace{4\frac{(1-2z)(1-3z)}{(z-0.5)(z-0.6)}}_{\substack{\text{“outer function”} \\ \text{(all poles and zeros outside } \mathbb{E})}}.$$

Definitions

An *inner function* is an H_∞ function with unit modulus almost everywhere in $\mathbb{T}$.

An *outer function* is an $f \in H_1$ that can be written as

$$f(z) = \alpha \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{z+e^{-i\omega}}{z-e^{-i\omega}} k(e^{i\omega})d\omega\right), \qquad z \in \mathbb{E},$$

where $k\colon \mathbb{T} \to \mathbb{R}$ is an integrable function, and $|\alpha| = 1$.

Remark: An outer function cannot have zeros in $\mathbb{E}$.

Inner-Outer Factorization (cont.)

Theorem (Beurling). Let $f \in H_1$ be nonzero. Then, $f = f_I \cdot f_O$, where f_I is inner and f_O is outer. This factorization is unique up to a constant of unit modulus.

Proof idea: Let $k = \log|f|$ (integrable by the lemma on slide 14) in the definition of outer function. □

Corollary (Riesz factorization theorem)

$f \in H_1$ iff there are $g, h \in H_2$ s.t. $f = gh$ and $\|f\|_{H_1} = \|g\|_{H_2} \|h\|_{H_2}$.

Proof. Since $f = f_I f_O$, where f_I is inner and f_O is outer, let $g = \sqrt{f_O}$ and $h = \sqrt{f_O} f_I$. □

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

Hankel Matrices and Operators

A causal discrete-time linear system G is defined by the relation

$$y_t = \sum_{k=0}^{\infty} g_k u_{t-k} = \sum_{k=-\infty}^{t} g_{t-k} u_k, \qquad t \in \mathbb{Z},$$

or, in matrix form,

$$\begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \ddots & & & 0 \\ g_1 & g_0 & 0 & \ddots & \\ \ddots & g_1 & g_0 & 0 & \\ & \ddots & g_1 & g_0 & \ddots \\ & & & g_1 & \ddots \end{bmatrix}}_{\substack{\text{Toeplitz form describing } G \\ \text{(infinite matrix, constant along its diagonals)}}} \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix}.$$

Hankel Matrices and Operators (cont.)

If we constrain the input $(u_t)_{t\in\mathbb{Z}}$ so that $u_t = 0$ for $t > 0$, and project $(y_t)_{t\in\mathbb{Z}}$ onto $\ell_2(\mathbb{Z}_+)$ (*i.e.*, only focus on y_t for $t \geqslant 0$), we obtain

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \\ g_2 & g_3 & \iddots & \\ \vdots & & & \end{bmatrix}}_{\substack{\text{infinite Hankel matrix} \\ \text{(constant along its anti-diagonals)}}} \begin{bmatrix} u_0 \\ u_{-1} \\ u_{-2} \\ \vdots \end{bmatrix}$$

Hankel operator, Γ_G*, with symbol* $G = \sum_{k=-\infty}^{\infty} g_k z^{-k}$, relating past inputs $u \in \ell_2(\mathbb{Z}_-)$ to future outputs $y \in \ell_2(\mathbb{Z}_+)$.

If R is the *reversion operator on* $L_2(\mathbb{T})$, $R\left(\sum_{k=-\infty}^{\infty} a_k z^{-k}\right) := \sum_{k=-\infty}^{\infty} a_{-k} z^{-k}$, and M_G is the multiplication operator on $L_2(\mathbb{T})$ by G, $M_G f = Gf$, then Γ_G can be seen as an operator on H_2:

$$\Gamma_G = P_{H_2} M_G R\Big|_{H_2}.$$

Hankel Matrices and Operators (cont.)

Note that if $G(z) = g_1 z^{-1} + g_2 z^{-2} + \cdots$ is the transfer function of a system described by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t, \end{aligned} \qquad \begin{array}{c} \text{State-space representation} \\ \text{(with state } x_t \in \mathbb{R}^n) \end{array}$$

then $G(z) = C(zI - A)^{-1}B$, and the Hankel matrix of $zG(z)$ is

$$\begin{bmatrix} g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \\ g_3 & g_4 & \iddots & \\ \vdots & & & \end{bmatrix} = \begin{bmatrix} CB & CAB & CA^2B & \cdots \\ CAB & CA^2B & CA^3B & \\ CA^2B & CA^3B & \iddots & \\ \vdots & & & \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}}_{\substack{\Psi_o : \mathbb{C}^n \to \ell_2 \\ \text{observability operator}}} \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}}_{\substack{\Psi_c : \ell_2 \to \mathbb{C}^n \\ \text{controllability operator}}}.$$

This means that the Hankel operator can be decomposed into a *controllability operator* (mapping past inputs to initial state x_0) and an *observability operator* (mapping the initial state to future outputs).

Norm of Γ_G

Assume that G is *controllable* and *observable*, *i.e.*, that Ψ_c is surjective and Ψ_o is injective, respectively. Since $\Gamma_G = \Psi_o \Psi_c$, we have, for every $x \in \ell_2$,

$$\|\Gamma_G x\|^2 = (\Gamma_G x, \Gamma_G x) = (\Psi_o \Psi_c x, \Psi_o \Psi_c x) = (\Psi_o^* \Psi_o \Psi_c x, \Psi_c x) = (\Psi_o^* \Psi_o y, y),$$

where $y = \Psi_c x$. Hence

$$\|\Gamma_G\|^2 = \sup_{\substack{y=\Psi_c x \\ \|x\|_{\ell_2} \leqslant 1}} (\Psi_o^* \Psi_o y, y) = \sup_{\substack{y=\Psi_c x \\ \|x\|_{\ell_2} \leqslant 1}} y^T [\Psi_o^* \Psi_o] y = \sup_{y^T [\Psi_c \Psi_c^*]^{-1} y \leqslant 1} y^T [\Psi_o^* \Psi_o] y.$$

The last step is due to that $y = \Psi_c x$ for some $x \in \ell_2$ s.t. $\|x\| \leqslant 1$ iff $y^T [\Psi_c \Psi_c^*]^{-1} y \leqslant 1$, which holds since $\min_{x \in \ell_2, y = \Psi_c x} \|x\|^2 = y^T [\Psi_c \Psi_c^*]^{-1} y$. This follows from a result in the bonus slides of Topic 8, which states that the minimizer x^{opt} satisfies $x^{\text{opt}} = \Psi_c^* z$ for some $z \in \mathbb{C}^n$ s.t. $y = \Psi_c \Psi_c^* z$, *i.e.*, $x^{\text{opt}} = \Psi_c^* [\Psi_c \Psi_c^*]^{-1} y$, hence $\|x^{\text{opt}}\|^2 = y^T [\Psi_c \Psi_c^*]^{-1} y$ (note that the assumption that $\mathscr{R}(\Psi_c) = \mathbb{C}^n$ holds because G is controllable).

Norm of Γ_G (cont.)

Now,

$$\begin{array}{l} L_c := \Psi_c \Psi_c^* = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k \\ L_o := \Psi_o^* \Psi_o = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k \end{array} \quad \text{are solutions of:} \quad \begin{array}{l} L_c - A L_c A^T = B B^T \\ L_o - A^T L_o A = C^T C. \end{array} \quad \textit{(Lyapunov equations)}$$

Therefore:

$$\begin{aligned} \|\Gamma_G\|^2 &= \max_{y^T L_c^{-1} y \leqslant 1} y^T L_o y && (x = L_c^{-1/2} y) \\ &= \max_{x^T x \leqslant 1} x^T L_c^{1/2} L_o L_c^{1/2} x && \textit{Easy eigenvalue problem} \\ &= \lambda_{\max}(L_c^{1/2} L_o L_c^{1/2}) \\ &= \lambda_{\max}(L_c L_o). \end{aligned}$$

Note. $\lambda_{\max}(AB) = \lambda_{\max}(BA)$, since $ABx = \lambda_{\max} x$ can be written as the set of equations $Ay = \lambda_{\max} x$, $Bx = y$, or equivalently, $BAy = \lambda_{\max} y$, and vice versa.

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

Nehari's Theorem

Notice that if $\Gamma = P_{H_2} M_g R\big|_{H_2}$ is a Hankel operator, then

$$\|\Gamma\| = \|P_{H_2} M_g R\| \leqslant \|P_{H_2}\| \|M_g\| \|R\| = \|g\|_\infty.$$

The following result establishes a deep connection between H_∞ problems and Hankel operators:

Theorem (Nehari)
If Γ is a bounded Hankel operator on H_2, then there is a $g \in L_\infty(\mathbb{T})$ s.t. $\Gamma = P_{H_2} M_g R\big|_{H_2}$, and $\|g\|_\infty = \|\Gamma\|$.

Remark: Two symbols $g, h \in L_\infty(\mathbb{T})$ give the same Hankel operator iff their nonnegative Fourier coefficients coincide, *i.e.*, $g(z) = \sum_{k=-\infty}^{\infty} g_k z^{-k}$ and $h(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}$, with $g_k = h_k$ for all $k \geqslant 0$. Thus, Nehari's theorem establishes the greatest lower bound on the ∞-norm of a $g \in L_\infty(\mathbb{T})$ whose projection onto H_2 is fixed.

Nehari's Theorem (cont.)

Corollary
Given $g \in L_\infty(\mathbb{T})$, we have that $\|\Gamma_g\| = \min_{h \in H_\infty^\perp} \|g - h\|_\infty$, where $H_\infty^\perp$ is the space of those $f(z) = \sum_{k=-\infty}^{-1} f_k z^{-k}$ which are analytic and bounded in $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$.

Given Γ, the problem of finding a symbol for Γ of minimum norm, *i.e.*,

$$\|\Gamma\| = \inf\{\|g\|_\infty \colon g \in L_\infty(\mathbb{T}) \text{ is a symbol for } \Gamma\},$$

is called the *Nehari extension problem*.

Proof of Nehari's theorem

We already know that if g is a symbol for Γ, then $\|\Gamma\| \leqslant \|g\|_\infty$. Our goal then is to show that there is a symbol for which we achieve equality. As the non-positive Fourier coefficients of g are fixed, we need to determine the positive ones, which amounts to extend Γ to a Hankel operator on L_2. We will do this by extending a related functional from H_1 to L_1.

The entries of the matrix of Γ are $a_{n+m} := (\Gamma z^{-n}, z^{-m}) = (\Gamma z^{-n-m}, 1)$. Therefore,

$$\left(\Gamma \sum_{n=0}^{N} b_n z^{-n}, \sum_{m=0}^{M} \overline{c_m} z^{-m}\right) = \left(\Gamma \sum_{n=0}^{N} b_n z^{-n} \sum_{m=0}^{M} c_m z^{-m}, 1\right).$$

Denote $\left(\sum_{m=0}^{M} c_m z^{-m}\right)^+ := \sum_{m=0}^{M} \overline{c_m} z^{-m}$. Then, for polynomials f_1, f_2 we can define the functional

$$\alpha(f_1 f_2) = (\Gamma f_1, f_2^+) = (\Gamma f_1 f_2, 1),$$

which satisfies $|\alpha(f_1 f_2)| \leqslant \|\Gamma\| \|f_1\|_2 \|f_2\|_2$.

Proof of Nehari's theorem (cont.)
By Riesz Factorization theorem, every $f \in H_1$ can be factorized as a product of H_2 functions f_1, f_2, and polynomials are dense in H_2, so α can be extended uniquely to $\tilde{\alpha}\colon H_1 \to \mathbb{C}$, by $\tilde{\alpha}(f) = \tilde{\alpha}(f_1 f_2) = (\Gamma f_1, f_2^+)$.

Furthermore, $|\tilde{\alpha}(f)| \leqslant \|\Gamma\| \|f_1\|_2 \|f_2\|_2 = \|\Gamma\| \|f\|_1$, so $\|\tilde{\alpha}\| \leqslant \|\Gamma\|$.

Since H_1 is a subspace of L_1, by Hahn-Banach there is an extension $\bar{\alpha}$ of $\tilde{\alpha}$ to L_1 s.t. $\|\bar{\alpha}\| = \|\tilde{\alpha}\| \leqslant \|\Gamma\|$.

Since the dual of $L_1(\mathbb{T})$ is $L_\infty(\mathbb{T})$, $\bar{\alpha}(f) = \displaystyle\int_{-\pi}^{\pi} f(e^{i\omega}) h(e^{i\omega}) d\omega$ for some $h \in L_\infty(\mathbb{T})$, with $\|h\|_\infty = \|\bar{\alpha}\| \leqslant \|\Gamma\|$. Now, for all $n, m \geqslant 0$,

$$a_{n+m} = (\Gamma z^{-n-m}, 1) = \bar{\alpha}(z^{-n-m}) = \int_{-\pi}^{\pi} e^{-i(n+m)\omega} h(e^{i\omega}) d\omega.$$

Therefore, $h(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}$ with $h_{-n} = a_n$ for all $n \geqslant 0$, and $\|h\|_\infty \leqslant \|\Gamma\|$.

This means that by taking $g(e^{i\omega}) = h(e^{-i\omega})$, we obtain the desired symbol for Γ. $\square$

Nehari's Theorem (cont.)

How can we compute the optimal symbol $g \in L_\infty(\mathbb{T})$?

Theorem (Sarason)
If Γ is a bounded Hankel operator on H_2, and $f \in H_2$ is nonzero and s.t. $\|\Gamma f\|_2 = \|\Gamma\|\|f\|_2$, then there is a unique symbol $g \in L_\infty(\mathbb{T})$ for Γ of minimum norm, $\|g\|_\infty = \|\Gamma\|$, and it is given by $g = \Gamma f / Rf$. Moreover, $|g(e^{i\omega})|$ is constant almost everywhere.

Proof. Let $g \in L_\infty(\mathbb{T})$ be s.t. $\|g\|_\infty = \|\Gamma\|$, and recall that $\Gamma f = P_{H_2} M_g R f$. Therefore,

$$\|\Gamma\|\|f\|_2 = \|\Gamma f\|_2 = \|P_{H_2} M_g R f\|_2 \leqslant \|M_g R f\|_2 \leqslant \|g\|_\infty \|Rf\|_2 = \|\Gamma\|\|f\|_2.$$

Since the leftmost and rightmost sides coincide, we have equality throughout. Therefore, $\|P_{H_2} M_g R f\|_2 = \|gRf\|_2$, *i.e.*, $gRf \in H_2$, so $\Gamma f = gRf$, or $g = \Gamma f / Rf$, which shows that g is unique. Moreover, since $\|gRf\|_2 = \|g\|_\infty \|Rf\|_2$, it follows that $|g(e^{i\omega})|$ is constant almost everywhere. $\square$

How can we find an $f \in H_2$ s.t. $\|\Gamma f\|_2 = \|\Gamma\| \|f\|_2$?

Let $y_0 \in \mathbb{R}^n$ achieve the maximum in $\|\Gamma_G\| = \max_{y^T L_c^{-1} y \leqslant 1} y^T L_o y$.
(*How?* Let $\tilde{y} = L_c^{-1/2} y$ and solve the eigenvalue problem: $\max_{\tilde{y}^T \tilde{y} \leqslant 1} \tilde{y}^T L_c^{1/2} L_o L_c^{1/2} \tilde{y}$.)

The sought f is s.t. $y_0 = \Psi_c f$, and to achieve equality in $\|\Gamma f\|_2 = \|\Gamma\| \|f\|_2$ it must have minimum norm. From the derivation at the end of Slide 21, this implies that

$$f = \Psi_c^* L_c^{-1} y_0,$$

or: $f_k = B^T (A^T)^k L_c^{-1} y_0$ for $k \geqslant 0$ (and zero otherwise), *i.e.*, $f(z) = zB^T (zI - A^T)^{-1} L_c^{-1} y_0$
Also, $\Gamma f(z) = (\Psi_o \Psi_c f)(z) = (\Psi_o y_0)(z) = \sum_{k=0}^{\infty} CA^k y_0 z^{-k} = zC(zI - A)^{-1} y_0$, so

$$\boxed{g(z) = \frac{(\Gamma f)(z)}{(Rf)(z)} = \frac{(\Psi_o y_0)(z)}{f(z^{-1})} = \frac{zC(zI - A)^{-1} y_0}{z^{-1} B^T (z^{-1} I - A^T)^{-1} L_c^{-1} y_0}.}$$

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

H_∞ Control Example

Consider the system: $$G(z) = \frac{z+2}{z-0.9}.$$

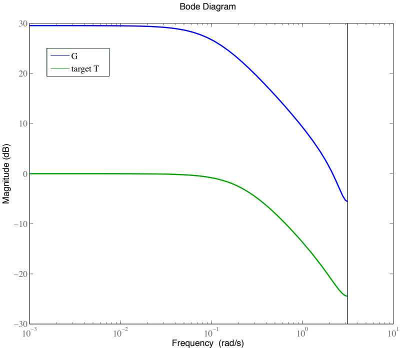
We want to control it so that the transfer function T from reference to output becomes

$$T(z) = \frac{1}{6.5}\,\frac{z+0.3}{z-0.8},$$

i.e., we want the closed loop to be slightly faster than G, and with static gain $T(e^{i0}) = 1$.

Using the Youla parameterization, we can impose these constraints by minimizing

$$\inf_{Q \in H_\infty} \|T - GQ\|_\infty.$$



Let's compute the optimum of

$$\inf_{Q\in H_\infty} \underbrace{\left\| \frac{1}{6.5}\,\frac{z+0.3}{z-0.8} - \frac{z+2}{z-0.9}Q(z) \right\|_\infty}_{=:J}.$$

Step 1: *Factorize poles and zeros in* $\mathbb{D}$

$$\begin{aligned}
\left\| \frac{1}{6.5}\,\frac{z+0.3}{z-0.8} - \frac{z+2}{z-0.9}Q(z) \right\|_\infty &= \left\| \frac{1+2z}{z+2}\left(\frac{1}{6.5}\,\frac{z+0.3}{z-0.8} - \frac{z+2}{z-0.9}Q(z) \right) \right\|_\infty \\
&= \left\| \frac{1}{6.5}\,\frac{(z^{-1}+0.3)(2z^{-1}+1)}{(z^{-1}-0.8)(z^{-1}+2)} - \tilde{Q}(z^{-1}) \right\|_\infty \\
&= \left\| -\frac{3}{104}\,\frac{(z+10/3)(z+2)}{(z-1.25)(z+0.5)} - \tilde{Q}(z^{-1}) \right\|_\infty,
\end{aligned}$$

where $\tilde{Q}(z) := \dfrac{1+2z}{z-0.9}Q(z)$.

Step 2: *Partial fraction expansion, to remove unstable poles*

$$
\begin{aligned}
-\frac{3}{104}\,\frac{(z+10/3)(z+2)}{(z-1.25)(z+0.5)} &\approx -0.0288+\frac{0.0701}{z+0.5}-\frac{0.2455}{z-1.25}\\
&= \underbrace{-0.0288+\frac{0.0701}{z+0.5}+0.1964}_{\in H_\infty}+\underbrace{\frac{-0.1964z}{z-1.25}}_{\in H_\infty^\perp}\\
&= \frac{0.1676z+0.1538}{z+0.5}-\frac{0.1964z}{z-1.25},
\end{aligned}
$$

so

$$
J=\left\|\frac{0.1676z+0.1538}{z+0.5}-Q'(z^{-1})\right\|_\infty,
$$

where $Q'(z):=\tilde{Q}(z)+\frac{0.1964z^{-1}}{z^{-1}-1.25}=\tilde{Q}(z)+\frac{0.1964}{1-1.25z}=\tilde{Q}(z)-\frac{0.1571}{z-0.8}$.

Step 3: *State-space realization of the problem*

$$
\frac{0.1676z+0.1538}{z+0.5}\,\frac{1}{z} \quad\Rightarrow\quad
\begin{aligned}
x_{k+1} &= \begin{bmatrix}-0.5 & 0\\ 1 & 0\end{bmatrix}x_k+\begin{bmatrix}0.5\\ 0\end{bmatrix}u_k\\
y_k &= \begin{bmatrix}0.3352 & 0.3077\end{bmatrix}x_k.
\end{aligned}
$$

Step 4: *Compute Gramians (by solving their Lyapunov equations)*

$$L_c = \begin{bmatrix} 0.3333 & -0.1667 \\ -0.1667 & 0.3333 \end{bmatrix}, \qquad L_o = \begin{bmatrix} 0.1385 & 0.1031 \\ 0.1031 & 0.0947 \end{bmatrix}.$$

Step 5: *Compute norm of Hankel matrix*

$$\|\Gamma\| = 0.1947.$$

Step 6: *Compute $f \in H_2$ s.t. $\|\Gamma f\|_2 = \|\Gamma\| \|f\|_2$*

$$y_0 = \begin{bmatrix} -0.3824 \\ -0.1834 \end{bmatrix}, \qquad f(z) = -0.94819 \frac{z + 0.7902}{z + 0.5}.$$

Step 7: *Compute optimal symbol of Hankel matrix*

$$(\Psi_o y_0)(z) = zC(zI - A)^{-1} y_0 = -0.18461 \frac{z + 0.7902}{z + 0.5},$$

so

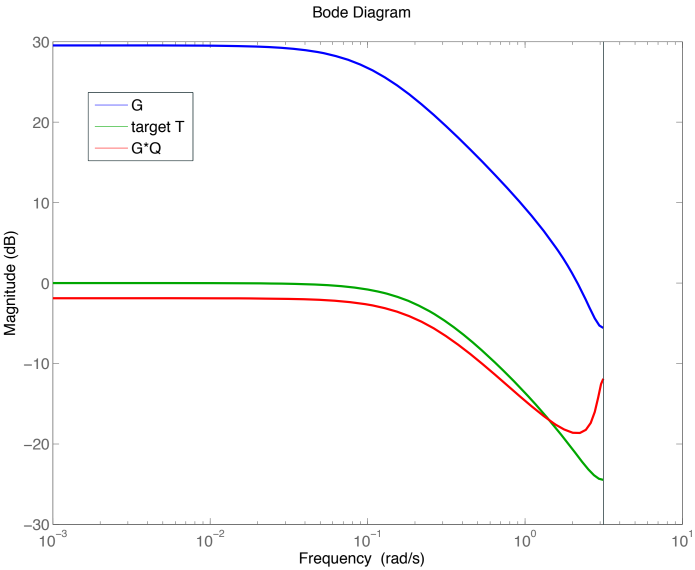
$$g(z) = 0.1750 \frac{(z + 0.7902)(z^{-1} + 0.5)}{(z^{-1} + 0.7902)(z + 0.5)}.$$

Notice that $|g(e^{i\omega})| = 0.1947$ for all ω (as we expected).

Step 8: *Compute optimal Q*

$$\begin{aligned} Q(z) &= \frac{z - 0.9}{1 + 2z} \left[\frac{0.1571}{z - 0.8} + \frac{0.1676z^{-1} + 0.1538}{z^{-1} + 0.5} - 0.1750 \frac{(z^{-1} + 0.7902)(z + 0.5)}{(z + 0.7902)(z^{-1} + 0.5)} \right] \\ &= \frac{0.096111(z - 0.9)}{(z + 0.7902)(z - 0.8)}. \end{aligned}$$

H_∞ Control Example (cont.)



Last Slide

Thank you for attending the course!

Outline

Feedback Control and Youla Parameterization

Approaches to H_∞ Control

The Big Picture

Hardy Spaces

Hankel Matrices and Operators

Nehari's Theorem

H_∞ Control Example

Bonus Slides

Bonus: Proof of Youla / Affine Parametrization

Notice that, in terms of the Youla parameter $Q := C/[1+GC]$,

$$
\begin{aligned}
T &= \frac{GC}{1+GC} = GQ \\
S &= \frac{1}{1+GC} = 1-GQ \\
S_i &= \frac{G}{1+GC} = G-G^2Q \\
S_u &= \frac{C}{1+GC} = Q,
\end{aligned}
$$

hence all sensitivity functions are affine in Q. Now, if G and Q are stable, all sensitivity functions are stable as well, while conversely, if the sensitivity functions are stable, $Q = S_u$ is stable too. $\square$

Bonus: Radial Limits of H_p Functions

Poisson representation

Consider an analytic $f : \bar{\mathbb{E}} \to \mathbb{C}$. By Cauchy's integral formula, for every analytic $h : \bar{\mathbb{E}} \to \mathbb{C}$:

$$f(z) = -\frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{f(w)}{w - z} dw = -\frac{1}{2\pi i} \oint_{\mathbb{T}} f(w) \left[\frac{1}{w - z} + h(w) \right] dw = -\frac{1}{2\pi} \oint_{\mathbb{T}} f(w) \left[\frac{w}{w - z} + w h(w) \right] \frac{dw}{iw},$$

for $z \in \mathbb{E}$, since the integral of an analytic function in $\bar{\mathbb{E}}$ around $\mathbb{T}$ is zero. Note that if $w = e^{it}$ $(t \in [-\pi, \pi])$, $dw/iw = dt$. We want to choose h so the formula in brackets is real. Now,

$$\frac{w}{w - z} + w h(w) = 1 + \frac{z}{w - z} + w h(w) = 1 + \frac{z\bar{w}}{1 - z\bar{w}} + w h(w), \qquad (w \in \mathbb{T})$$

so we can force $w h(w) = \overline{z\bar{w}/(1 - z\bar{w})} = \bar{z} w/(1 - \bar{z} w)$, or $h(w) = \bar{z}/(1 - \bar{z} w)$. Then, making $w = e^{it}$ and $z = re^{i\theta}$ $(r > 1)$, we obtain

$$f(re^{i\theta}) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2\mathrm{Re}\left(\frac{re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} \right) \right] f(e^{it}) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\frac{r^2 - 1}{1 - 2r\cos(\theta - t) + r^2}}_{=:P(r, \theta - t) \text{ “Poisson kernel in } \mathbb{E}\text{”}} f(e^{it}) dt.$$

Poisson representation of H_p functions ($p > 1$)
Note first that, for every $\alpha \in (1, \infty)$,

$$f(\alpha r e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) f(\alpha e^{it}) dt \qquad (r \in (1, \infty), \theta \in [-\pi, \pi]).$$

To see this, apply the Poisson representation to $f_\alpha(z) = f(\alpha z)$, which is also analytic in $\bar{\mathbb{E}}$.

If $f \in H_p$ for $p > 1$, then $\tilde{f}_\alpha \in L_p[-\pi, \pi]$, where $\tilde{f}_\alpha(\omega) := f_\alpha(e^{i\omega})$, and $\|\tilde{f}_\alpha\|_p \leqslant \|f\|_p$. Consider a sequence $(\tilde{f}_{\alpha_n})$ where $\alpha_n \to 1_+$. Since $L_p = L_q^*$, where q is s.t. $1/p + 1/q = 1$, by Banach-Alaoglu, there is a subsequence $(\tilde{f}_{\alpha_k})$ s.t. $\tilde{f}_{\alpha_k} \to g \in L_p$ in a weak* sense (see bonus slides of Topic 7). Thus,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) g(t) dt = \frac{1}{2\pi} \langle P(r, \theta - \cdot, g \rangle = \lim_{k \to \infty} \frac{1}{2\pi} \langle P(r, \theta - \cdot, \tilde{f}_{\alpha_k} \rangle = \lim_{k \to \infty} f(\alpha_k r e^{i\theta}) = f(r e^{i\theta}),$$

since f is continuous in $\mathbb{E}$; this yields a Poisson representation for analytic functions in $\mathbb{E}$.

Fatou's Theorem. Let $g \in L_1[-\pi,\pi]$, and assume that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)g(t)dt, \qquad \text{for all } r \in (1,\infty), \theta \in [-\pi,\pi].$$

Then, the *radial limit* $\lim_{r\to 1_+} f(re^{i\theta}) = g(\theta)$ exists for almost all $\theta \in [-\pi,\pi]$.

Proof. From the Poisson representation of $f \equiv 1$, $\int_{-\pi}^{\pi} P(r,\theta-t)dt = 2\pi$ for all r,θ. Then, by integration by parts, if $G(t) := \int_{-\pi}^{t} g(\tau)d\tau$,

$$f(re^{i\theta}) - g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)[g(t)-g(0)]dt = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial P(r,\theta-t)}{\partial t}[G(t)-g(\theta)t]dt.$$

Now, for $0 < \delta \leqslant |\theta - t| \leqslant \pi$,

$$\left|\frac{\partial P(r,\theta-t)}{\partial t}\right| \leqslant \frac{2r(r^2-1)}{[1-2r\cos(\delta)+r^2]^2} \to 0 \text{ as } r \to 1_+,$$

while $-\dfrac{1}{2\pi}\displaystyle\int_{\theta-\delta}^{\theta+\delta} \frac{\partial P(r,\theta-t)}{\partial t}[G(t)-g(\theta)t]dt = -\frac{1}{2\pi}\int_{0}^{\delta} \frac{\partial P(r,t)}{\partial t} t\left[\frac{G(\theta+t)-G(\theta-t)}{2t} - g(\theta)\right]dt.$

Given $\varepsilon > 0$, let $\delta > 0$ be small enough so $|g(\theta) - [G(\theta+t)-G(\theta-t)]/2t| \leqslant \varepsilon$ for all $t \in [0,\delta]$ (this holds for almost all θ, by the Radon-Nikodym theorem). These two estimates imply that $\lim_{r\to 1_+} f(r) = g(0)$. $\square$

Hardy's theorem

Let $f: \mathbb{E} \to \mathbb{C}$ be analytic, and define $M_p(f;r) := \left[(2\pi)^{-1} \int_{-\pi}^{\pi} |f(re^{it})|^p \, dt\right]^{1/p}$ for $r \in (1,\infty)$ and $p \in [1,\infty]$. Then, $M_p(f;r)$ is non-increasing in r.

Proof (Taylor, 1950). Let us define a function $F: \mathbb{E} \to L_p[-\pi,\pi]$ by $[F(z)](\theta) = f(ze^{i\theta})$ $(\theta \in (-\pi,\pi))$. Notice that $\|F(z)\|_p = M_p(f,|z|)$. We will show now that the maximum of $\|F(z)\|_p$ over the open region $r\mathbb{E} = \{z \in \mathbb{C}: |z| > r\}$ cannot be achieved inside $r\mathbb{E}$, unless $\|F(z)\|_p$ is constant in it. Indeed, if $\|F(z_0)\|_p = \sup_{z\in r\mathbb{E}} \|F(z)\|_p$ for some $z_0 \in r\mathbb{E}$, then since by Cauchy's integral formula (defining the integral entry-wisely)

$$[F(z_0)](\theta) = f(z_0 e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 e^{i\theta} + \delta e^{i(\theta+t)}) dt = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} F(z_0 + \delta e^{it}) dt\right](e^{i\theta}),$$

where $\delta > 0$ is small enough so that the integration path is inside $r\mathbb{E}$, and it includes points z for which $\|F(z)\|_p < \|F(z_0)\|_p$, then $\|F(z_0)\|_p \leqslant \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \|F(z_0 + \delta e^{it})\|_p \, dt \leqslant \|F(z_0)\|_p$, which contradicts the assumption that $\|F(z)\|_p$ is not constant in the integration path. This contradiction proves that $M_p(f;r) = \sup_{z\in r\mathbb{E}} \|F(z)\|_p$ is non-decreasing in r. $\square$

Bonus: Radial Limits of H_p Functions (cont.)

The previous three results imply that every $f \in H_p$, for $p > 1$, has the Poisson representation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)\tilde{f}(t)dt = f(re^{i\theta}),$$

where $\tilde{f}(t) = \lim_{r\to 1_+} f(re^{it})$ for all $t \in [-\pi,\pi]$. Furthermore, since $\|\tilde{f}_\alpha\|_p \leqslant \|f\|_p$, the Lebesgue dominated convergence theorem implies that $\|\tilde{f}\|_p = \|f\|_p$.

Note. Our approach to the development of a Poisson representation fails for H_1 functions because $L_1[-\pi,\pi]$ is not the dual of any normed space. In particular, for an $f \in H_1$, using the Riesz representation theorem for the dual of $C[-\pi,\pi]$, one arrives at the *Poisson-Stieltjes representation*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)dG(t) = f(re^{i\theta}),$$

where $G \in \mathrm{NBV}[-\pi,\pi]$, but extra effort is needed to show that it is differentiable (which leads to the *F. and M. Riesz theorem*).